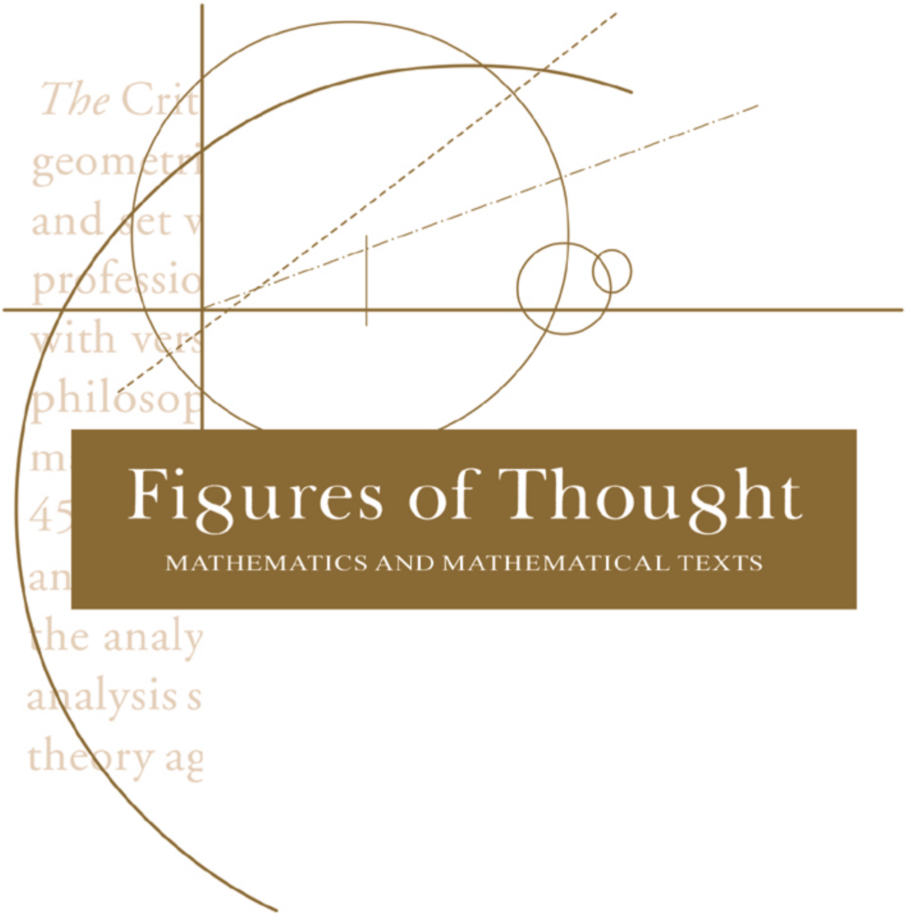

Figures of Thought

MATHEMATICS AND MATHEMATICAL TEXTS

David Reed

ROUTLEDGE

Also available as a printed book
see title verso for ISBN details

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Mathematics and Mathematical Texts

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London and New York

First published 1995
by Routledge
11 New Fetter Lane, London EC4P 4EE

This edition published in the Taylor & Francis e-Library, 2005.

"To purchase your own copy of this or any of Taylor & Francis or Routledge's collection of thousands of eBooks please go to www.eBookstore.tandf.co.uk."

Simultaneously published in the USA and Canada by Routledge
29 West 35th Street, New York, NY 10001

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British Library Cataloguing in Publication Data
A catalogue record for this book is available from the British Library

Library of Congress Cataloging in Publication Data
A catalog record for this book is available from the Library of Congress

ISBN 0-203-16828-3 Master e-book ISBN

ISBN 0-203-26347-2 (Adobe eReader Format)
ISBN 0-415-08146-7 (Print Edition)

To Zena …without whom not…

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Preface

> Day unto day uttereth speech and night unto night declareth knowledge…
>
> Ps. XIX

> But the contemplative life is somehow above the level of humanity…
>
> Nic. Eth. X vii

This book is the product of reflection and research undertaken over a period of twenty years under a wide variety of circumstances. It has its origins in the simple questions that David Smigelskis, Charles Wegener and Eugene Garver asked and taught me to ask at the University of Chicago two decades ago and the pleasure that accompanies the attempt to answer such questions has not diminished since. The influence of the ideas of Richard McKeon will also be evident to anyone who is familiar with them. More recently the kind and encouraging words of Stuart Shenker provided the impetus to undertake the task of combining these thoughts into a systematic whole.

Over these decades there has been a steady increase in the number of analyses of works from the ‘scientific literature’ and the notion that such texts can be treated *as texts* has become much more widespread. Readers will surely not be surprised to find in their hands an analysis which includes not only writings of Euclid and Descartes but also more modern and indeed contemporary mathematicians as well. The mathematical literature contains many wonderful examples from a range of eras and in a variety of genres. Increased awareness of the value of this literature both within and without the mathematical community is the principal objective to which this book is devoted.

Special thanks are owed to the Mathematical Institute, Angus Macintyre, Bryan Birch, Aldo and Gigi and all the others who have made my re-entry into the mathematical community so enjoyable.

Introduction

Mathematical argument has received comparatively little attention over the years compared with the voluminous literature devoted to ‘philosophy of mathematics’ and ‘foundations of mathematics’. Although all mathematics is set forth in texts which argue, explicitly or implicitly, for their own version of ‘doing mathematics’ and in spite of exhortations from all sides to read and study mathematics in the ‘original’ the general view remains that mathematical facts and mathematical subject matters exist somehow independently of the texts in which they are expressed.

My purpose is to demonstrate the utility of analysing mathematical texts *as texts and as arguments* and to suggest the ways in which the addition of this new dimension to the existing corpus of thought about mathematics can illuminate some of the problems and conundrums facing philosophers of mathematics while raising some new issues to be addressed. At the same time ‘working mathematicians’ may be able to increase the cogency and coherence of their thought through an increased awareness of the ways and means by which it can be organized and presented.

The approach put forward is not a version of ‘philosophy of mathematics’ in that it does not seek the philosophical principles that may or may not ground mathematics and its methods, nor does it attempt to codify the consequences which philosophers should derive from mathematical facts for matters which are not strictly mathematical, such as the nature of thought or truth. Furthermore, in spite of the emphasis on ‘texts’, I shall not discuss questions of style, presentation or overall ‘philosophical orientation’, at least in so far as such discussions would pertain to giving form to pre-existing content.

With few models of similar analysis to refer to, the *possibility* of discoursing on mathematical argument as well as its utility must be demonstrated. With this dual objective in view the following essay is divided into three parts, each of which provides an example of the type of analysis being promoted as well as reflections on the relationships between this and other kinds of mathematical thought and discourse.

In Part I the first book of Euclid's *Elements* is carefully reviewed to provide an exegesis which takes into account the complex turns and twists of Euclid's argument. Although many thousands of pages have been devoted to the *Elements* since its (unknown) date of first publication, few interpreters have sought to follow Euclid as he establishes his subject matter and shapes the argument of his treatise. Much of what will be said here is therefore at variance with established patterns of interpretation of the *Elements*. To put the Euclidean approach into perspective, Book I of the *Elements* is compared with René Descartes' *Géométrie* and David Hilbert's *Grundlagen der Geometrie,* two of the most significant texts on the subject of elementary geometry to have been written since Euclid. The point of comparison is the nature of the geometric subject matters which are established by each author.

In Part II the books of the *Elements* treating ratio and number (Books V–X) are examined. In these sections of his treatise, which are considerably less elementary than Book I, Euclid sets the material he has previously developed into new contexts through definitions of 'magnitude' and 'number'. These are topics which have been of great concern to philosophers of mathematics since the nineteenth century when various programmes to provide a 'foundation' for mathematics were put forward. The methods that mathematicians have employed to characterize number systems and produce new types of measure can be directly contrasted with Euclid's use of contexts which neither describe nor construct material but provide new ways of exploring previously developed topics. The contrast is brought out most forcefully by examining the work of two mathematicians from the second half of the nineteenth century, Richard Dedekind and Leopold Kronecker, as they developed the foundations of modern number theory. Because these two mathematicians were (and are) considered to represent two extremes of mathematical thought and method, it is particularly illuminating to see how similar their approaches are when contrasted with Euclid's methods.

In Part III the closing books of the *Elements* (Books XI–XIII) dealing with solid figures are analysed. Euclid brings his argument to a close by finding a type of 'completeness' which arises from the subject matter itself. In doing so he does not foreclose the posing of additional questions or the postulating of new areas for research, but rather demonstrates how the particular subject matter and method which he has adopted form a complete system and provide their own sense of closure. These are matters which have been of concern to contemporary mathematicians as they have sought to formulate and reformulate languages in which to express results of higher and higher generality. While the issues are of importance to working mathematicians (and have even been the cause of polemic within the mathematical community) they have drawn relatively little attention from philosophers and writers on mathematics. The work of André Weil and Alexandre Grothendieck in reformulating algebraic geometry is a case in point. Weil's 'universal domains' and Grothendieck's 'categories' and 'universes' are attempts to encompass the broadest possible range of objects and results in a single system. The issues which arise are not simply those of 'technical mathematics' (although their discussion requires more mathematical background than in Parts I and II) and comparison with Euclid brings out the common problematic.

In a brief concluding chapter a few of the philosophical issues brought out in the analysis will be reviewed and commented upon.

The method adopted here is 'inductive' in nature, moving from the analysis of particular examples to broader views of mathematical argument. This is in keeping with an enterprise which seeks to explore the general nature of mathematical argument in the particular arguments of mathematicians. None the less, it may be useful to compare the conclusions adumbrated by the examples analysed here with some of the issues arising in contemporary reflection on mathematics.

Euclid's text forms the 'backbone' of the analysis not because of its relatively elementary nature or the extent of its influence but rather because it demonstrates a wider variety of approaches to mathematical argument and a higher degree of awareness of the issues involved than any other single text in the history of mathematics. This is not to claim a superior position for it but merely to indicate the nature of the utility that a close reading of this text can have for all mathematicians and friends of mathematics.

However, what follows can by no means be considered a full-scale interpretation of the *Elements*. Many critical issues in the text have been ignored as their discussion would not fit in with the overall objective of the argument. A more comprehensive review of Euclid's work and related matters in Greek science and philosophy must await another occasion.

Part I

The subject matter of geometry in Euclid, Descartes and Hilbert

1
The opening of the *Elements*

1.1
Orientation

A vast amount of commentary has accumulated since ancient times around the Definitions, Postulates and Common Notions with which Euclid commences the *Elements*. In certain cases (one thinks immediately of the notorious fifth or ‘Parallel’ Postulate) much of the subsequent history of mathematics can be viewed as one extended commentary on this text! For the English reader, Heath’s edition of the *Elements* surveys much of this commentary up to its publication date (2nd edn, 1925) and has the benefit of being easily available.[1] The analysis conducted below does not require an extensive familiarity with this material as it refers to the lines of commentary that have been generally pursued only to clarify the points at which it diverges from the various traditional approaches. At certain points, however, the analysis does turn on specific words or phrases in the Greek original. Where this is the case the appropriate philological or linguistic background is provided.

The reader may find the nature of the commentary more in line with the methods which might be more typically employed in the analysis of a poem or other ‘literary’ texts rather than in the examination of a mathematical treatise. The feeling of disorientation that results is likely to be productive. To facilitate the initial stage of the analysis the content of the first seven Definitions of the *Elements* (together with Definitions 1 and 2 of Book XI) are laid out schematically in Table 1.1.

1.2
Points and lines

Euclid’s Definition 1[2], ‘A point is that which has no part’, frames the entire sequence of Definitions and provides a ‘principle’ in the sense of ‘a beginning’.[3] Euclid does not have reference here either to spatial

Table1.1

Object	*Definition*	*Limits*	*Regularity*
Point	That which has no part	None	None
Line	Breadthless length	Points	Straight lines lie evenly with points on themselves
Surface	Length and breadth only	Lines	Plane surfaces lie evenly with straight lines upon themselves
Solid	Length and breadth and depth	Surfaces	None

location or to 'quantity' as such. It will be seen in Part II that quantities are derived terms in the *Elements,* arising from specific ways to analyse specific things. Instead of these more frequently found approaches, Euclid employs the term 'part' (Greek μέρος) as he will again at other turning points in his argument, and a negative grammatical construction (the only other Definition to use negation in Book I being the last Definition, that of parallel lines, which shares a framing function with this Definition) to indicate a limit or an extremity.

The limit in question is not a limit of magnitude or position, rather it is a limit of intelligibility, an extreme of discourse. In this opening phase of the *Elements* 'parts' are things in terms of which other things can be defined, discussed and understood. That which has *no part* merely has existence—completely undifferentiated existence. One does not have to construct or imagine such things; anything can be considered to be a point if it is considered without parts.[4] All such things are undifferentiated: one cannot speak of singular and plural, of here and there, or indeed of any other type of qualification.[5] Hence this definition marks the limit of that which can be discussed or analysed. Everything else which will enter into the argument of the *Elements* will have 'parts' of some kind, and determining the nature of these parts and their relationships to each other and to the wholes which they constitute is the method of Euclid's science. Euclid's choice of this rather strange formulation for defining 'point' will be seen to be reflective of his entire approach.

It should be superfluous to add that Euclid is not here attempting to describe (well or poorly) the visual appearance of a point, nor his, our or the ideal mathematician's intuition or perception of what a point is. Any reading of Euclid along these lines (Heath's notes contain a number of examples) omits all of the aspects of this definition which differentiate it from other similar definitions used by mathematicians ancient and modern. While it may have been traditional then as now to begin geometry texts with the definition of a point, whatever the tradition demanded Euclid has provided a definition which provides its own justification for commencing his argument. It is sufficient to note that he begins his argument with that which indicates the beginning of discourse; to refer to extraneous reasons for the formulation or positioning of this definition is unnecessary.

The orientation provided by Definition 1 clarifies the sequence which follows. Unlike a point, a line does have parts, one part in particular which is labelled length.[6] By referring to something with one part Euclid shows clearly that he does not have in mind a sense of 'part' which has to do with division or with 'sub-objects'. Parts are the ways in which things may be known or described. As far as their definition is concerned, lines are things about which one can know only one thing, their length. For convenience this type of definition will be referred to as definition by specification of a measureable (in this case length). It is inappropriate to think of length in this context in geometric, metric or measure theoretic terms. All of these approaches require previous specification of some type of measure or some kind of line. Definition 2 merely states that lines (a) are distinguished from points by having parts, (b) are distinguished from other geometric things by having only one part and (c) can be compared amongst themselves by this part, their length. To repeat, anything can be considered to be a line, provided it is considered as having only one characteristic or term of comparison. Clearly there is no distinguishing (at this stage in the argument) between 'two' lines of the 'same' length. The grammar of singular and plural is limited precisely to distinctions of length.

It is also inappropriate to think of this definition as embodying some (primitive) type of 'dimensional' analysis. As can be seen from Table 1.1, each item in the sequence 'point', 'line', 'surface', 'solid' is defined in a (slightly) different manner. The distinguishing characteristic of dimensional definitions is that the things to be defined are defined in *the same way except for* the number of dimensions involved. Euclid's sequence by contrast has a form or shape and each term occupies the place it does for a particular reason. It therefore cannot have a 'dimensional' character, even a primitive one. The *Elements* opens with the definition of point because point is the extreme of discourse. It continues with lines and surfaces because these have parts through which they can be known. Eventually we will see why it terminates with solids.

Now if lines are those things which differ from one another in precisely one way how does one actually go about making comparisons between lines? The answer is provided by Definition 3 which, as far as most commentators are concerned, does not define anything at all. This Definition states that points are the extremities[7] of lines, i.e. the comparisons of lines as lengths is effected by means of points as their extremities. One might say therefore that Definition 3 defines *delimited lines* by specifying the means by which this delimitation takes place. There are two aspects to this:

- points can function in this way without further specification just because there is nothing further to say about a point other than that it is a point;
- by functioning as the extremities of lines, points acquire a further characteristic which allows them to be differentiated, i.e. one can now speak of singular and plural, point and points.

Once again it should be clear that Euclid is not attempting to describe how lines and points 'look', nor is he asserting that lines are 'made up of points' or foreshadowing some notion of 'incidence'. The purpose of these Definitions is to define terms and establish a subject matter, not to describe already known or existing things.

Definition 4 is of 'straight lines' and is, on most readings of the *Elements,* virtually unintelligible. The approach proposed here, however, suggests a simple and clear interpretation. As points define the delimiting of lines, straight lines are precisely those for which no additional specification of the relationship between points-as-extremities and lines-as-delimited is either necessary or possible. For straight lines this relationship is always the same. No metric or measure theoretic specification is implied here. The delimitation happens in the same manner throughout the line, but nothing is said about what this manner may be.[8]

Definitions 2, 3 and 4 taken together provide the paradigm in Euclid's mathematics of a 'measured thing':

- that which is measured (in this case lines) is defined in terms of a measurable (in this case length);
- the measurable is further determined by specifying how the measuring or delimitation is to be performed (in this case by points-as-extremities) *and this specification requires nothing further* (there is nothing further to say about points);
- a special type or kind of measured thing (in this case straight lines) is defined by requiring that the delimitation of the measurable does not vary with what is delimited and that no further specification of this relationship is necessary.

This must be understood as a formal procedure which moves mechanically step-by-step to define and determine the measured thing. Once the starting point (!) is adopted there is no opportunity for variation or deviation. This is Euclid's approach to 'rigour' in mathematics and failure to understand his method leads to the view that some 'selection' process based on tradition or philosophical inclination led Euclid to pick the particular formulations and sequence of definitions that we find in the *Elements*.[9] The importance of a clear understanding of the nature of Euclid's argument goes beyond an appreciation of the clarity and cogency of his thinking; it is fundamental to understanding his subject matter. As has been shown, points, lines and straight lines can be completely and satisfactorily related to one another through the definitions themselves. Because of this they do not pose by themselves interesting mathematical problems for Euclid and do not constitute the subject matter of the *Elements*. None of this can be said for surfaces.

1.3
Surfaces

Definition 5 of surfaces mimics Definition 2 of lines with two parts, length and width, instead of length alone. The remarks above on the definition of line carry over with no substantial modifications. Clearly, once again, spatial intuition is not at issue. Nor should it be thought that a specific combination of length and breadth in a notion of area is referred to. The two measurables, length and breadth, exist independently and, at least at this stage of the argument, there is no way to put them together.

Definition 6 mimics Definition 3 in specifying lines as the extremities of surfaces as points are of lines. However, the grammar of the plural 'lines' differs from that of points. As noted above, it was precisely in functioning as the extremities of lines that any distinctions between points could be drawn. From their definition alone there are no characteristics which would permit either numerical or generic distinctions among them. But lines do have a variety of ways in which the singular/plural distinction can be applied:

- lines can differ by length,
- some lines can be distinguished as straight lines,
- lines can act as the extremities of surfaces (allowing distinctions among lines of the same length).

Now in delimiting surfaces, lines act as extremities in that they limit the double comparison of length and breadth to one term. This does not imply or involve a 'picture' of lines 'cutting' surfaces but simply a formal relating of definitions to each other. There is no reason in this